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QUADRATIC AND CUBIC TRANSITION ELEMENTS

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Based on the investigations of Barsoum,[1] Henshell and Shaw,[2] quarterpoint quadratic elements have been successfully used as crack tip elements in fracture mechanics. This concept of singular element was extended to cubic isoparametric elements.[3] Recently it was discovered by Lynn and Ingraffea[4] that under special configuration, transitional elements improve the accuracy of stress intensity factor computations. In this report, we have obtained

(CONT'D ON REVERSE)

20. ABSTRACT (Cont'd)

the location of mid-side nodes of these transitional elements for the quadratic as well as cubic elements. The cubic transitional elements were used for the double-edge crack problem, and it was found that there was improvement in accuracy for a configuration which consisted only of singular and transitional elements. However, for a well laid out grid, the improvement was only marginal.

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INTRODUCTION

Based on the investigations of Barsoum,[1] Henshell and Shaw,[2] quarter-point quadratic elements have been successfully used as crack tip elements in fracture mechanics. This concept of singular element was extended to cubic isoparametric elements.[3] Recently it was discovered by Lynn and Ingraffea[4] that under special configuration, transitional elements improve the accuracy of stress intensity factor computations. These transitional elements are located in the immediate vicinity of the singular elements with the mid-side nodes adjusted as to reflect or extrapolate the square root singularity on the stresses and strains at the tip of the crack.

In this report, we have obtained the locations of mid-side nodes of these transitional elements for the quadratic as well as cubic elements. Explicit computations for a typical element are symbolically carried out using MACSYMA*.[5] These computations reveal that in addition to the desired square root singularities, the crack tip senses a stronger singularity, i.e., of order one. Further, the strength of this singularity cannot be controlled, as was possible for the cubic and quadratic collapsed elements, where, by tying the collapsed nodes together, we could easily abolish this strong singularity.

These cubic elements also have Hibbit-type[6] singularities. The locations of mid-side nodes for these singularities have also been determined.

References are listed at the end of this report.

*MACSYMA is a large program for symbolic manipulation at MIT.

The cubic transitional elements were used for a double-edge crack problem, and it was found that there was improvement in accuracy for a configuration which consisted only of singular and transitional elements. However, for a well laid out grid, the improvement was only marginal. MACSYMA has proved to be an indispensable tool for the present investigation.

SECTION I

Consider a quadratic quadrilateral isoparametric element,

$$x = \sum_{i=1}^{8} N_i X_i \quad , \quad y = \sum_{i=1}^{8} N_i Y_i \tag{1}$$

$$u = \sum_{i=1}^{8} N_i U_i \quad , \quad v = \sum_{i=1}^{8} N_i V_i \tag{2}$$

where N_i are the shape functions of 'Serendipity' family,[6] and are given by,

$$\text{CORNER NODES} : \quad N_1 = \frac{1}{4}(1-\xi)(1-\eta)(-\xi-\eta-1) \quad , \quad \text{etc.} \tag{3}$$

$$\text{MID-SIDE NODES} : \quad N_5 = \frac{1}{2}(1-\xi^2)(1-\eta) \quad , \quad \text{etc.} \tag{4}$$

Without loss of generality consider the sectorial element, together with the mapped unit element in the transformed plane, shown in Figure 1. For simplicity, considering the one-dimensional case along line 1-2 in Figure 1 (i.e., $\eta = -1$) we have from (1)

$$x = \frac{1}{2}\xi(\xi-1) + \frac{1}{2}\xi(1+\xi)L + (1-\xi^2)\beta L \tag{5}$$

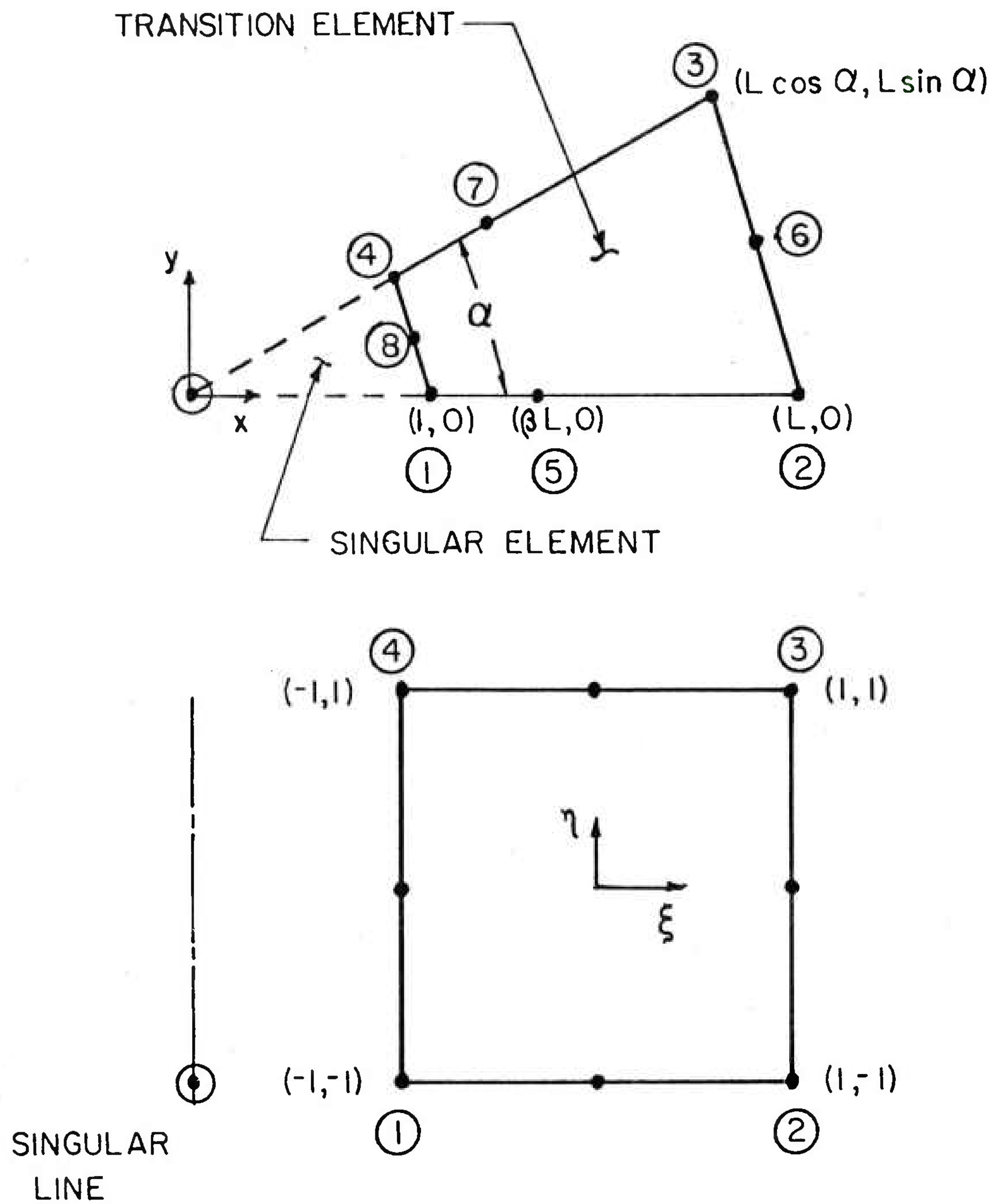


FIGURE 1. QUADRATIC QUADRILATERAL ISOPARAMETRIC ELEMENT AS TRANSITION ELEMENT

The condition for the coalescence of roots of (5) at $x = 0$, together with the condition that $\beta L > 1$ gives

$$\beta L = \frac{L+2\sqrt{L}+1}{4} \tag{6}$$

This is the result, in a slightly different form, obtained by Lynn and Ingraffea.[4] With this location of mid-side nodes, the mapping of the general element of Figure 1 becomes, from (1) and (2),

$$x = \frac{1}{8}\{(\eta+1)\cos\alpha + (1-\eta)\}\{\xi(\sqrt{L}-1) + (\sqrt{L}+1)\}^2 \tag{7}$$

$$y = \frac{1}{8}(\eta+1)\{\xi(\sqrt{L}-1) + (\sqrt{L}+1)\}^2 \sin\alpha \tag{8}$$

The Jacobian of the transformation (1) and (2) is then given by

$$J = \frac{\partial(x,y)}{\partial(\xi,\eta)} = \frac{1}{16}(\sqrt{L}-1)\{\xi(\sqrt{L}-1) + (\sqrt{L}+1)\}^3 \sin\alpha \tag{9}$$

As can be seen from (7), (8), and (9), the Jacobian has a third order zero while x and y have second order zeroes at

$$\xi = -\frac{\sqrt{L}+1}{\sqrt{L}-1} \tag{10}$$

Using the inverse of the Jacobian matrix, the strain component can be written as

$$\frac{\partial u}{\partial x} = \frac{1}{J}\left\{\frac{\partial u}{\partial \xi}\frac{dy}{d\eta} - \frac{\partial u}{\partial \eta}\frac{dy}{d\xi}\right\} \tag{11}$$

Substituting the various derivatives and collecting terms we get

$$\frac{\partial u}{\partial x} = \frac{A_1}{(\xi(\sqrt{L}-1) + \sqrt{L}+1)^2} + \frac{A_2}{(\xi(\sqrt{L}-1) + \sqrt{L}+1)} + A_3 \tag{12}$$

where A_1, A_2, and A_3 are given in Appendix A.

Comparing (12) with (7) and (8) it is seen that the strain component not only has singularity of order one half but also of order one. Similarly we have

$$\frac{\partial u}{\partial y} = \frac{1}{J}\left\{-\frac{\partial u}{\partial \xi}\frac{dx}{d\eta} + \frac{\partial u}{\partial \eta}\frac{dx}{d\xi}\right\} = \frac{A_4}{(\xi(\sqrt{L}-1) + \sqrt{L}+1)^2} + \frac{A_5}{(\xi(\sqrt{L}-1) + \sqrt{L}+1)} + A_6 \tag{13}$$

where A_4, A_5, A_6 are given in Appendix A.

SECTION II

Consider now the cubic, 12-node, quadrilateral isoparametric element,

$$x = \sum_{i=1}^{12} N_i X_i \quad , \quad y = \sum_{i=1}^{12} N_i Y_i \tag{14}$$

and displacements

$$u = \sum_{i=1}^{12} N_i U_i \quad , \quad v = \sum_{i=1}^{12} N_i V_i \tag{15}$$

where the shape functions are given by

$$\text{CORNER NODES}: \quad N_1 = \frac{1}{32}(1-\xi)(1-\eta)\{9(\xi^2+\eta^2)-10\} \quad , \quad \text{etc.} \tag{16}$$

$$\text{MID-SIDE NODES}: \quad N_2 = \frac{9}{32}(1-3\xi)(1-\xi^2)(1-\eta) \quad , \quad \text{etc.} \tag{17}$$

The general transitional element together with its map in ξ-η plane is given in Figure 2. For simplicity consider the one-dimensional case along line 1-2-3-4 (i.e., $\eta = -1$),

$$x = \frac{1}{16}\left(\xi^3(-9+27\beta_1L-27\beta_2L+9L) + \xi^2(9-9\beta_1L-9\beta_2L+9L) + (1-27\beta_1L+27\beta_2L-L) + (-1+9\beta_1L + 9\beta_2L-L)\right) \tag{18}$$

The requirement that (18) be quadratic in ξ, together with the condition of coalescence of roots gives the following, physically possible solution for locations of mid-side nodes for all L,

$$\beta_1L = \frac{L+4\sqrt{L}+4}{9}, \qquad \beta_2L = \frac{4L+4\sqrt{L}+1}{9} \tag{19}$$

With the above values the general mapping of the element shown in Figure 2 then becomes

$$x = \frac{1}{8}\left((\eta+1)\cos\alpha - (\eta-1)\right)\{\xi(\sqrt{L}-1) + (\sqrt{L}+1)\}^2 \tag{20}$$

$$y = \frac{1}{8}(\eta+1)\{\xi(\sqrt{L}-1) + (\sqrt{L}+1)\}^2 \sin\alpha \tag{21}$$

and the Jacobian of the transformation becomes

$$J = \frac{\partial(x,y)}{\partial(\xi,\eta)} = \frac{1}{16}(\sqrt{L}-1)\left(\xi(\sqrt{L}-1) + (\sqrt{L}+1)\right)^3 \sin\alpha \tag{22}$$

These expressions are the same as for quadratic elements (compare eqs. (7), (8), and (9)), and hence the Jacobian has third order zeroes and x,y have second order zeroes, at

$$\xi = -\frac{\sqrt{L}+1}{\sqrt{L}-1} \tag{23}$$

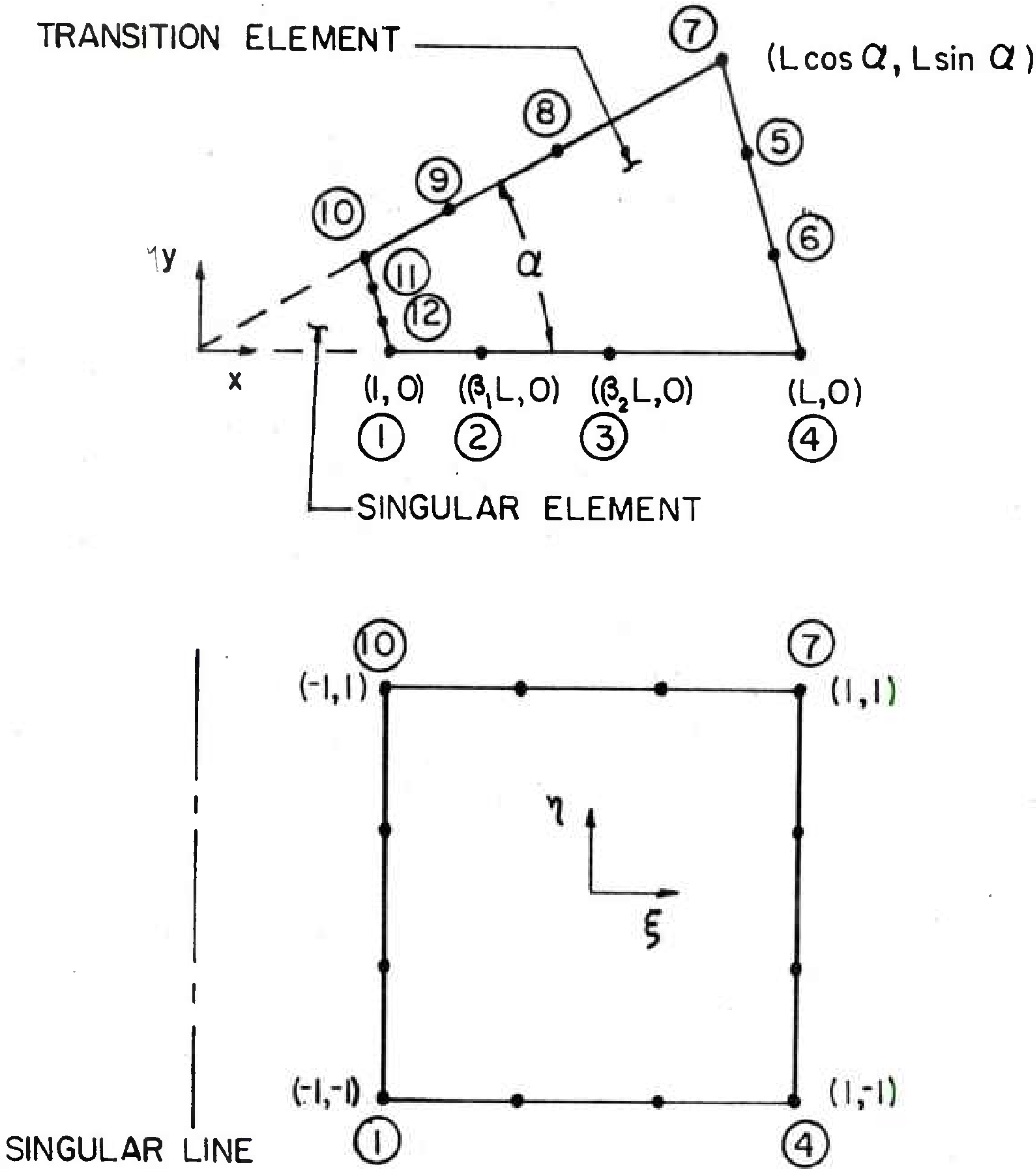


FIGURE 2. CUBIC QUADRILATERAL ISOPARAMETRIC ELEMENT AS TRANSITION ELEMENT

Following the procedure outlined before, the strain components can be obtained from the following

$$\frac{\partial u}{\partial x} = \frac{B_1}{(\xi(\sqrt{L}-1) + \sqrt{L}+1)^2} + \frac{B_2}{(\xi(\sqrt{L}-1) + \sqrt{L}+1)} + B_3 , \tag{24}$$

$$\frac{\partial u}{\partial y} = \frac{B_4}{(\xi(\sqrt{L}-1) + \sqrt{L}+1)^2} + \frac{B_5}{(\xi(\sqrt{L}-1) + \sqrt{L}+1)} + B_6 , \tag{25}$$

where B_1 through B_6 are given in Appendix A. Similar expressions hold for derivatives of v. Equations (24) and (25) again reveal the same kinds of singularities as (12) and (13).

SECTION III

In the cubic elements there is an additional set of locations of mid-side nodes which give Hibbit-type[6] singularity. This is obtained from the condition that all the three roots of (18) coalesce. The location of nodes is given by

$$\begin{aligned} \beta_1 L &= \left(\frac{L^{1/3} + 2}{3}\right)^3 \\ \beta_2 L &= \left(\frac{2L^{1/3} + 1}{3}\right)^3 \end{aligned} \tag{26}$$

and the transformations become

$$\begin{aligned} x &= \frac{1}{16} \{(\eta+1)\cos\alpha - (\eta-1)\}\{\xi(L^{1/3}-1) + L^{1/3}+1\}^3 \\ y &= \frac{1}{16} \{(\eta+1)\sin\alpha\}\{\xi(L^{1/3}-1) + L^{1/3}+1)\}^3 \end{aligned} \tag{27}$$

and the Jacobian becomes

$$J = \frac{\partial(x,y)}{\partial(\xi,\eta)} = \frac{3\sin(\alpha)}{128} (L^{1/3}-1)\{\xi(L^{1/3}-1) + L^{1/3}+1\}^5 \tag{28}$$

Following the procedure outlined before, it can be shown that

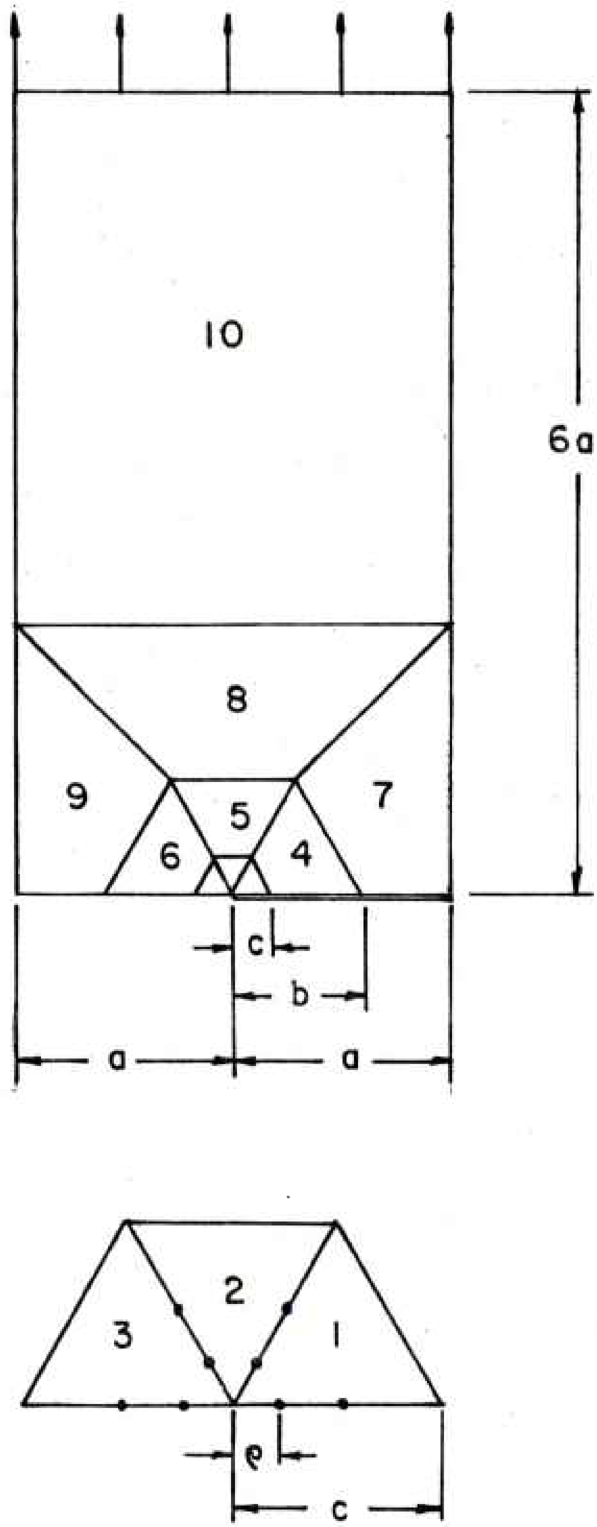
$$\frac{\partial u}{\partial x} = \frac{C_1}{(\xi(L^{1/3}-1) + L^{1/3}+1)^3} + \frac{C_2}{(\xi(L^{1/3}-1) + L^{1/3}+1)^2} +$$

$$+ \frac{C_3}{(\xi(L^{1/3}-1) + L^{1/3}+1)} + C_4 \tag{29}$$

The above equation indicates that in this case the singularities are of order 1, 2/3, and 1/3. This combination is of no immediate interest in linear fracture in homogeneous media.

SECTION IV

The sample problem of a double-edge cracked plate of Ref. 4 was selected for numerical assessment of transition elements when used with 12-node collapsed singular elements. Figure 3 is an idealization we usually take for such a mode I crack problem. The distance ρ between the crack tip and the nearest node in a collapsed element is often taken in the range of 0.5% to 3% of the crack length a. The ratios a/b and b/c are usually in the range of 2 to 10. Stress intensity factors for several values of ρ, b/c, and a/b with and without the use of transition elements are tabulated in Table I. Comparing to the reference value, $K_I = \sigma\sqrt{\pi a}F(a/2a)$, where F(1/2) = 1.184,[7] the percentage errors $\Delta\%$ are also shown in the table. The result with the use of transition elements is better only when a very large ratio of b/c (=20) is used.



a = 1.8

c = 9ρ

FIGURE 3. AN IDEALIZATION FOR A QUARTER OF A DOUBLE-EDGE CRACKED PLATE

TABLE I. STRESS INTENSITY FACTOR AND PERCENTAGE ERROR FOR A DOUBLE-EDGE CRACKED PLATE USING 12-NODE COLLAPSED SINGULAR ELEMENTS WITH AND WITHOUT TRANSITION ELEMENTS. FINITE ELEMENT IDEALIZATION OF FIGURE 3.

			Without Transition Elements		With Transition Elements	
ρ	b/c	a/b	SIF	Δ%	SIF	Δ%
0.005	4	10	2.8808	2.31	2.8736	2.06
	10	4	2.8376	0.78	2.7831	-1.16
	20	2	2.9863	6.06	2.7851	-1.09
0.01	4	5	2.7986	-0.61	2.7926	-0.82
	10	2	2.8334	0.63	2.7813	-1.22

TABLE II. STRESS INTENSITY FACTOR AND PERCENTAGE ERROR FOR A DOUBLE-EDGE CRACKED PLATE USING 12-NODE COLLAPSED SINGULAR ELEMENTS WITH AND WITHOUT TRANSITION ELEMENTS. FINITE ELEMENT IDEALIZATION OF FIGURE 4.

		Without Transition Elements		With Transition Elements	
ρ	a/c	SIF	Δ%	SIF	Δ%
0.005	40	3.325	18.09	2.7658	-1.77
0.01	20	2.963	5.23	2.7654	-1.79
0.02	10	2.8115	-0.15	2.7650	-1.80
0.04	5	2.7632	-1.86	2.655	-5.71

Another idealization, Figure 4, similar to the one used by Lynn and Ingraffea[4] was used to recompute stress intensity factors for various values of a/c to see whether the transition elements in cubic isoparametric elements can give as good improvement in accuracy as reported in Ref. 4 in the quadratic isoparametric case. These results are tabulated in Table II. It shows again the result obtained from the use of transition elements is better only when a very large ratio of a/c is used.

In this report the stress intensity factors were calculated from the normal component of displacement of the node on the crack surface and nearest to the crack tip. It usually gives better results than the average value computed from nodal displacements along the rays from the crack tip at various angles.[8]

For elastic crack problems, the correct order of singularity at the crack tip is taken care of by the collapsed singular elements. The use of transition elements does not practically improve the accuracy.

CONCLUSIONS

In this report we have been able to obtain explicit expressions for singularities the crack tip senses from a transitional element. The application of these elements for a few practical problems of fracture mechanics as well as stress concentration factors has been partially successful. It is believed this is because the crack tip senses not only the square root singularity, but also a stronger singularity. The strength of this singularity cannot be controlled as was possible for collapsed singular elements, where the strong singularity was essentially eliminated by tying the nodes together.

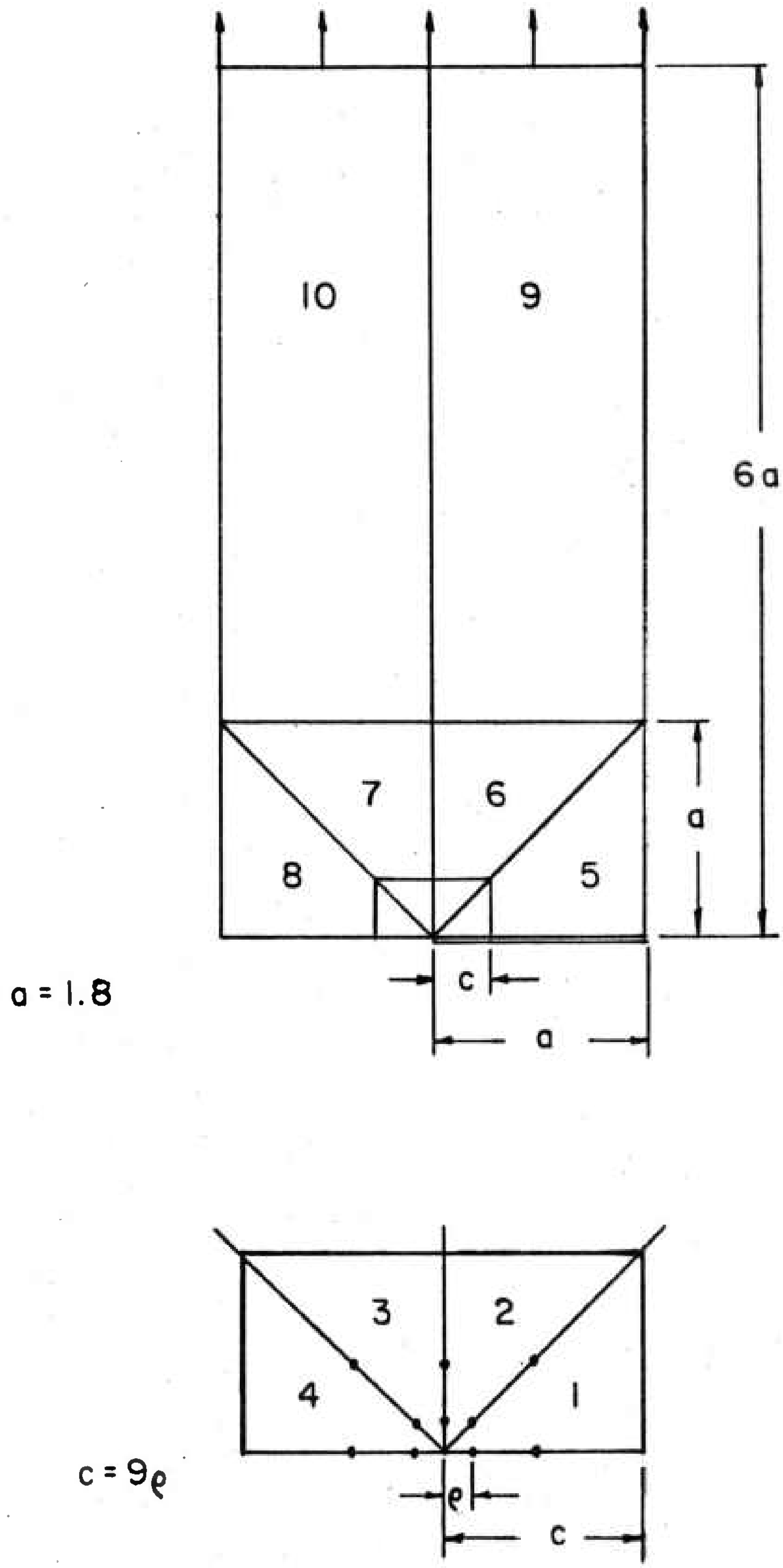


FIGURE 4. A SIMILAR IDEALIZATION USED IN [4] FOR A QUARTER OF A DOUBLE-EDGE CRACKED PLATE

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APPENDIX A

In this appendix we give the explicit expressions for the coefficients of the various terms in the strain components given in the text.

$$R = \sqrt{L}$$

$$A_1 = \frac{2(\eta+1)}{(R-1)^2}\{4\eta(R-1)[Ru_8-u_6] + 4R[u_7-u_5]$$

$$+ (2\eta R+R-2\eta+1)[u_2-Ru_4] + (2\eta R-R-2\eta-1)[u_3-Ru_1]\}$$

$$A_2 = -\frac{1}{2(R-1)^2}\{2(3\eta^2+4\eta+1)(R-1)(u_8-u_6) + 4(\eta+1)(R+1)u_7$$

$$- 4(\eta+3)(R+1)u_5 + (R(3\eta^2+7\eta+4) - 3\eta(\eta+1))u_4$$

$$+ (3R\eta(\eta+1) - (3\eta^2+7\eta+4))u_3 + (R(3\eta^2+5\eta+4) - (3\eta^2+\eta-8))u_2$$

$$- (R(3\eta^2+\eta-8) - (3\eta^2+5\eta+4))u_1\}$$

$$A_3 = -\frac{2}{(R-1)^2}(2u_5-u_2-u_1)$$

$$A_4 = -\frac{2((\eta+1)\cos\alpha - (\eta-1))}{(R-1)^2\sin\alpha}\{4R\eta(R-1)u_8 + 4R(u_7-u_5) - 4\eta(R-1)u_6$$

$$- (R(2\eta+1) - (2\eta-1))(Ru_4-u_2) + (R(2\eta-1) - (2\eta+1))(u_3-Ru_1)\}$$

$$A_5 = \frac{1}{2(R-1)^2\sin\alpha}\{2(R-1)[\cos\alpha(3\eta^2+4\eta+1) - (3\eta^2-4\eta+1)](u_8-u_6)$$

$$+ 4(R+1)(\cos\alpha(\eta+1) - (\eta-3))u_7 - 4(R+1)(\cos\alpha(\eta+3) - (\eta-1))u_5$$

$$- (\cos\alpha[R(3\eta^2+7\eta+4) - 3\eta(\eta+1)] - [R(3\eta^2-\eta-8) - (3\eta^2-5\eta+4)])u_4$$

$$+ (\cos\alpha[3R\eta(\eta+1) - (3\eta^2+7\eta+4)] - [R(3\eta^2-5\eta+4) - (3\eta^2-\eta-8)])u_3$$

$$+ (\cos\alpha[R(3\eta^2+5\eta+4) - (3\eta^2+\eta-8)] + [-3R\eta(\eta-1) + (3\eta^2-7\eta+4)])u_2$$

$$- (\cos\alpha[R(3\eta^2+\eta-8) - (3\eta^2+5\eta+4)] + [-R(3\eta^2-7\eta+4) + 3\eta(\eta-1)])u_1\}$$

$$A_6 = -\frac{2}{(R-1)^2 \sin\alpha}\left\{\cos\alpha(-2u_5+u_2+u_1) + (2u_7-u_4-u_3)\right\}$$

$$B_1 = \frac{\eta+1}{(R-1)^3}\left\{\frac{1}{4}\,[R^2(27\eta^2-18\eta-1) - R(54\eta^2-36\eta-38) + 27\eta^2-18\eta-1](Ru_1-u_4)\right.$$

$$+\, 9R(2R+1)(u_9-u_2) + 9R(R+2)(u_3-u_8)$$

$$+\,\frac{9}{4}\,(R-1)^2(9\eta^2-2\eta-3)(u_5-Ru_{12}) - \frac{9}{4}\,(R-1)^2(9\eta^2+2\eta-3)(u_6-Ru_{11})$$

$$\left.+\,\frac{1}{4}\,[R^2(27\eta^2+18\eta-1) - R(54\eta^2+36\eta-38) + 27\eta^2+18\eta-1](u_7-Ru_{10})\right\}$$

$$B_2 = \frac{1}{(R-1)^3}\left\{-\frac{1}{16}\,[R^2(45\eta^3+27\eta^2-\eta+105) - R(90\eta^3+54\eta^2-146\eta-222)\right.$$

$$+\, 45\eta^3+27\eta^2-37\eta-3]u_1 + \frac{9}{4}\,(2R^2+6R+1)[(\eta+3)u_2 - (\eta+1)u_9]$$

$$-\,\frac{9}{4}\,(R^2+6R+2)[(\eta+3)u_3 - (\eta+1)u_8] +$$

$$+\,\frac{1}{16}\,[R^2(45\eta^3+27\eta^2-37\eta-3) - R(90\eta^3+54\eta^2-146\eta-222) +$$

$$+\, 45\eta^3+27\eta^2-\eta+105]u_4$$

$$-\,\frac{9}{16}\,(R-1)^2(\eta+1)[(15\eta^2-7)u_5 + (15\eta^2+6\eta-5)u_{11}]$$

$$+\,\frac{9}{16}\,(R-1)^2(\eta+1)[(15\eta^2+6\eta-5)u_6 + (15\eta^2-7)u_{12}]$$

$$-\,\frac{1}{16}\,(\eta+1)[R^2(45\eta^2+36\eta-1) - 2R(45\eta^2+36\eta-37) + 45\eta^2+36\eta+35]u_7$$

$$\left.+\,\frac{\eta+1}{16}\,[R^2(45\eta^2+36\eta+35) - 2R(45\eta^2+36\eta-37) + 45\eta^2+36\eta-1]u_{10}\right\}$$

$$B_3 = \frac{9}{2}\,\frac{1}{(R-1)^3}\,\{(2R+1)u_1 - (5R+4)u_2 + (4R+5)u_3 - (R+2)u_4\}$$

$$B_4 = \frac{\{(\eta+1)\cos\alpha - \eta + 1\}}{\sin\alpha(R-1)^3}\,\{\frac{1}{4}\,[R^2(27\eta^2-18\eta-1) - R(54\eta^2-36\eta-38) +$$

$$+ 27\eta^2-18\eta-1](-Ru_1+u_4) + 9R(2R+1)(u_2-u_9) +$$

$$+ 9R(R+2)(-u_3+u_8) + \frac{9}{4}\,(R-1)^2(9\eta^2-2\eta-3)(-u_5+Ru_{12})$$

$$+ \frac{9}{4}\,(R-1)^2(9\eta^2+2\eta-3)(u_6-Ru_{11}) +$$

$$+ \frac{1}{4}\,[R^2(27\eta^2+18\eta-1) - R(54\eta^2+36\eta-38) + 27\eta^2+18\eta-1](-u_7+Ru_{10})\}$$

$$B_5 = \frac{1}{\sin\alpha(R-1)^3}\,\{\frac{1}{16}\,[R^2(45\eta^3(\cos\alpha-1) + 27\eta^2(\cos\alpha+3) - \eta(\cos\alpha+71)$$

$$+ 35(3\cos\alpha+1)) + R(90\eta^3(-\cos\alpha+1) - 54\eta^2(\cos\alpha+3) +$$

$$+ 2\eta(73\cos\alpha-1) + 74(3\cos\alpha+1)) + 45\,\eta^3(\cos\alpha-1) +$$

$$+ 27\eta^2(\cos\alpha+3) - \eta(37\cos\alpha+35) - 3\cos\alpha-1]u_1$$

$$- \frac{9}{4}\,((\eta+3)\cos\alpha-\eta+1)[(2R^2+6R+1)u_2 - (R^2+6R+2)u_3]$$

$$- \frac{1}{16}\,[R^2(45\eta^3(\cos\alpha-1) + 27\eta^2(\cos\alpha+3) - \eta(37\cos\alpha-35)$$

$$- (3\cos\alpha+1)) + R(90\eta^3(-\cos\alpha+1) - 54\eta^2(\cos\alpha+3)$$

$$+ 2\eta(73\cos\alpha-1) + 74(3\cos\alpha+1)) + 45\eta^3(\cos\alpha-1) +$$

$$+ 27\eta^2(\cos\alpha+3) - \eta(\cos\alpha+71) + 35(3\cos\alpha+1)]u_4$$

$$+ \frac{9}{16}\,(R-1)^2(15\eta^3(\cos\alpha-1) + 3\eta^2(5\cos\alpha+7) - \eta(7\cos\alpha+1)$$

$$- 7\cos\alpha-5)u_5 - \frac{9}{16}\,(15\eta^3(\cos\alpha-1) + 3\eta^2(7\cos\alpha+5)$$

$$+ \eta(\cos\alpha+7) - 5\cos\alpha-7)(R-1)^2 u_6 +$$

$$+ \frac{1}{16} [R^2(45\eta^3(\cos\alpha-1) + 27\eta^2(3\cos\alpha+1) + \eta(35\cos\alpha+37)$$

$$- (\cos\alpha+3)) + R(90\eta^3(-\cos\alpha+1) - 54\eta^2(3\cos\alpha+1)$$

$$+ 2\eta(\cos\alpha-73) + 74(\cos\alpha+3)) + 45\eta^3(\cos\alpha-1) +$$

$$+ 27\eta^2(3\cos\alpha+1) + \eta(71\cos\alpha+1) + 35(\cos\alpha+3)]u_7$$

$$+ \frac{9}{4} ((\eta+1)\cos\alpha-\eta+3)[-(R^2+6R+2)u_8 + (2R^2+6R+1)u_9] -$$

$$- \frac{1}{16} [R^2(45\eta^3(\cos\alpha-1) + 27\eta^2(3\cos\alpha+1) + (71\cos\alpha+1) +$$

$$+ 35(\cos\alpha+3)) + R(90\eta^3(-\cos\alpha+1) - 54\eta^2(3\cos\alpha+1)$$

$$+ 2\eta(\cos\alpha-73) + 74(\cos\alpha+3)) + 45\eta^3(\cos\alpha-1) +$$

$$+ 27\eta^2(3\cos\alpha+1) + \eta(35\cos\alpha+37) - (\cos\alpha+3)]u_{10} +$$

$$+ \frac{9}{16} (R-1)^2[15\eta^3(\cos\alpha-1) + 3\eta^2(7\cos\alpha+5) + \eta(\cos\alpha+7)$$

$$- (5\cos\alpha+7)]u_{11} -$$

$$- \frac{9}{16} (R-1)^2[15\eta^3(\cos\alpha-1) + 3\eta^2(5\cos\alpha+7) - \eta(7\cos\alpha+1)$$

$$- (7\cos\alpha+5)]u_{12}$$

$$B_6 = \frac{9}{2\sin\alpha(R-1)^3} \{(2R+1)[-\cos\alpha u_1 + u_{10}] + (5R+4)[\cos\alpha u_2 - u_9]$$

$$- (4R+5)[\cos\alpha u_3 - u_8] + (R+2)[\cos\alpha u_4 - u_7]\}$$

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